Propositional Logic Review II

Proofs and Simple Rules of Inference

So far we've looked at one method of demonstrating that something is true (or follows from certain premises) in FOL—truth tables. However, there are limitations to what truth tables can show us. They cannot, for example, show that it follows from 'All men are mortal' that 'Socrates is mortal'. We are going to turn to a different way of demonstrating truth and consequence in FOL: proof. The format that we use for proofs in this course is Fitch Format. Proofs in fitch format look something like this:

1	$P \vee Q$	
2	$\neg Q$	
3	$\quad P$	
4	$\quad P$	R, 3
5	$\quad Q$	
6	$\quad \neg Q$	R, 2

There is a vertical line, to the left of which is the number of each line in the proof. A horizontal line separates the premises from the inferential steps (which we will get to in a bit). Sometimes we will have proofs with no premises - in which case nothing will appear above the horizontal line. Lines that are premises or temporary assumptions do not need any justification, but all other steps require the rule of inference you are using, and which line you are applying the inference to (which appears to the right).

Simple Rules of Inference:

Rules of inference allow us to go from certain assumptions to conclusions that logically follow from those assumptions. Simple rules of inference take one step. Keep in mind that you can *only* use these rules to the outermost connective. Here are several simple rules:

Conjunction Introduction ($\wedge$ Intro)
$P, Q \vdash P \wedge Q$

Conjunction Elimination ($\wedge$ Elim)
$P \wedge Q \vdash P, P \wedge Q \vdash Q$

Disjunction Introduction($\vee$ Intro)
$P \vdash P \vee Q$

Negation Elimination($\neg$ Elim)
$\neg\neg P \vdash P$

Conditional Elimination ($\rightarrow$ Elim)
$P \rightarrow Q, P \vdash Q$

Biconditional Elimination ($\leftrightarrow$ Elim)
$P \leftrightarrow Q, P \vdash Q, P \leftrightarrow Q, Q \vdash P$

Contradiction Introduction ($\perp$ Intro)
$P, \neg P \vdash \perp$

Contradiction Elimination ($\perp$ Elim)
$\perp, \vdash P$

Example: Using fitch format and simple rules of inference, prove the following:

1. $(P \wedge \neg\neg Q) \vdash Q$

2. $(P \wedge Q) \wedge R \vdash Q \wedge R$

3. $\neg\neg P \vdash P \vee Q$

4. $\neg\neg(P \wedge Q) \wedge (R \wedge S) \vdash (P \vee Q) \wedge S$

5. $P \wedge \neg\neg Q, Q \rightarrow \neg\neg R, \vdash R \wedge P$

6. $P \wedge Q, P \leftrightarrow \neg\neg R, \vdash R$

7. $(P \wedge Q) \leftrightarrow (Q \rightarrow R), Q \wedge \neg\neg P \vdash R \vee S$

Complex Rules of Inference:

Some rules of inference take multiple steps. For these rules, we make temporary assumptions in the middle of our proof.

Negation Introduction (Reduction ad Absurdum) ($\neg$ Intro)

Sometimes we want to prove that a negation is true (e.g. $\neg(P \wedge \neg P)$). In these cases we temporarily assume that the negatum ($P \wedge \neg P$) is true and then show that a contradiction follows. After this we can safely conclude that the negation holds. For this sort of inference we site the original assumption and each line of the contradiction in our proof. After the sub-proof, we cannot use any lines that occurred within the subproof.

Disjunction Elimination ($\vee$ Elim)

When dealing with a disjunction of the form $P \vee Q$, we obviously do not know which disjunct—P or Q is correct. But for some inferences it doesn't matter. Sometimes, the same thing, R is going to follow from both P and from Q, so we can safely conclude R on the basis of the disjunction. To show this, we need two subproofs. In the first subproof we suppose that P is correct and show that R follows, and in the second subproof we suppose that Q is correct and show that R follows. In the second subproof we cannot site any lines that appeared int he first, and after both subproofs we cannot site any line that occurred in either. In the 'justification' section of disjunction elimination, site the original disjunction, the line with both suppositions, and both lines with the appropriate conclusion.

Example: Using fitch format and both simple and complex rules of inference, prove the following:

8. $P \wedge Q \vdash \neg(\neg P \wedge \neg Q)$

9. $\neg P \vee \neg Q \vdash \neg(P \wedge Q)$

Conditional Introduction (→ Intro)

To introduce a conditional of the form $P \to Q$ we must begin a subproof where we temporarily suppose that P is true. We then continue the subproof until we reach a line reading Q. We then close the subproof and conclude $P \to Q$—citing the all lines within the subproof. As with disjunction elimination and negation introduction, we cannot cite any lines within the subproof after we close it.

Biconditional Introduction (↔ Intro)

Introducing a biconditional requires two subproofs. In order to prove that $P \leftrightarrow Q$, we first temporarily suppose that P and continue the proof until we reach Q. We then close the first subproof, and, the a second subproof, temporarily suppose that Q and continue the proof until we reach P. We then close the second subproof and conclude $P \leftrightarrow Q$. When completing the second subproof, we cannot cite any lines within the first. And, after concluding the second subproof, we cannot cite any lines occurring within either subproof.

Example: Using fitch format and both simple and complex rules of inference, prove the following:

10. $P \to Q \vdash \neg Q \to \neg P$

11. $(P \to Q) \vee Q \vdash P \to Q$

12. $P \wedge \neg\neg Q \vdash P \leftrightarrow Q$

Practice Problems:

Using simple and complex rules of inference, prove each of the following:

13. $\neg(P \rightarrow Q) \wedge \neg(Q \rightarrow R) \vdash S$

14. $P \leftrightarrow Q \vdash (\neg P) \leftrightarrow (\neg Q)$