Proof Bootcamp

Tips and Tricks for Proofs

Sometimes, when you see a proof, you might not know where to start. Don't worry—you usually don't need to know how the whole proof fits together to get started. Even if you can just outline the overall form of the proof, you'll get partial credit. Often, the outline can help you figure out your next step. Here is a guide (really just rules of thumb) to help you approach problems. It wont work for everything, but should help for most problems you see on exams.

1. **Break Apart Premises**
   If you have premises that are conjunctions, use conjunction elimination to split them apart in your first few steps. Sometimes you won't need one (or more) of the conjuncts. But separating out the conjuncts is never incorrect, and it often gives you the most tools possible in your proof. If one premise is a conditional and the other is the antecedent, start by deriving the consequent. The same goes for biconditional elimination and negation elimination—get the simple steps out the way quickly.

2. **Consider Your Conclusion:**
   In many ways, your conclusion is the most important piece of information that you're given. Very often it tells you what the overall structure of the proof will be.
   If it's a conjunction, you'll probably use conjunction introduction in your last step. Prove each conjunct separately and then put them together at the very end.
   If it's a negation, you can bet that the overall proof will include negation introduction. You may not see all the steps, but you can provide the outline - write down a subproof with the negatum (your conclusion minus the negation sign) as the premise in the subproof, write your conclusion below the subproof and the contradiction sign as the last line within your subproof. Now you know you have to find a contradiction.
   If it's a disjunction, things get a little tricky. Sometimes, you'll be able to prove one of the disjuncts and conclude the overall disjunction with disjunction introduction. Other times, it isn't the case that either disjunct follows from your premises—only the disjunction as a whole does. In this case, your best bet is to try negation introduction. Start by assuming your disjunction is false, and

derive a contradiction. Then conclude the double negation of your disjunction, and use negation elimination.

If it's a material conditional, odds are the overall form will be conditional introduction. Start a subproof where the antecedent (P in $P \rightarrow Q$) appears as a premise and the consequence (Q in $P \rightarrow Q$) appears as the last line in the subproof, with the final line as your conclusion. Now you know that you have to find a way to prove the consequent.

If it's a biconditional, you'll probably need to use biconditional introduction. You'll have two subproofs - each of which looks like the subproof for the material conditional. If you are proving something of the form $P \leftrightarrow Q$ you'll start by assuming P and get to Q and then start by assuming Q and get to P.

If it's a sentence symbol you don't see anywhere in the premises, the premises are probably inconsistent. Try to locate the inconsistency in the premises.

3. Consider Your Premises:

So you've figured out part of the proof from your conclusion. It's time to think about your premises. Usually, you are going to eliminate connectives that you see in the premises. We've already taken care of conjunctions, but that leaves several connectives left. If you have a conditional, you'll probably need to use conditional elimination. If you already have formulas of the form P and $P \rightarrow Q$ you can do this in one step. If not - try to derive P. What steps would you need to use to derive it? If you have a biconditional, the same applies (but keep in mind that you could either infer P from Q or Q from P.

If your premise is a disjunction you'll probably need to use disjunction elimination; that requires more subproofs. For each disjunct, start a subproof with the disjunct as a premise. Remember: the conclusion of each subproof will have the be the exact same thing. If have no other ideas the conclusion of your argument (or the last line of your subproof if you're disjunction elimination occurs within another subproof) is a good bet.

4. Reason Informally:

Hopefully you've gotten at least some of the proof sketched out by now. Even if you haven't, you know what your starting points are and what your conclusion is. Try to reason informally and write down your informal reasoning on the side of the page. Often times, your informal reasoning is capturing a really solid formal argument. So after you've written down your informal argument, try to mirror it with the formal rules of proof that you have at your disposal. Usually the formal argument takes a few more steps than your informal argument, but it should give you a good idea of what to shoot for partway through your proof.

5. Construct a Truth-Table:

If you're still confused, try constructing a truth-table for the premises and conclusion. Often, the truth table can inform you of what intermediate steps to prove. If your premises tautologically entail some other sentence, that's a big hint that the sentence appears somewhere in your proof. Try to derive that sentence, then work from there to your conclusion.

6. Try a Proof by Contradiction:

So - everything else has failed. You've tried looking at your premises, your conclusion, have reasoned informally and constructed a truth-table. Nothing seems to be working. Don't panic. It's probably time to try a proof by contradiction. Maybe your conclusion *isn't* a negation - but still, you're going to use negation introduction. So if your conclusion is P (where P can really stand for anything) start a subproof where you assume that $\neg P$ and get a contradiction. Then, use negation introduction to derive $\neg\neg P$ and negation elimination to conclude P.

7. Repeat 1-6:

You might not have the whole proof yet, but you should have made some progress. Odds are, by now you've started at least one subproof. That gives you a new premise and a new conclusion. Repeat the above steps for the new (temporary) premise and conclusion. Break apart conjunctions, consider your conclusion, consider your new premise (in combination with premises you already had) reason informally and try a proof by contradiction again.

8. Remember: There are lots of ways to complete proofs correctly. Don't let indecision between possible strategies stop you from making progress. Several things that you're considering might work—pick one and give it a try.

Advanced Rules of Inference

Previously, we've discussed two types inferences for each logical connective. There's a rule for introducing a conjunction and eliminating a conjunction, a rule for introducing a negation and eliminating a negation, etc. The good news is that those rules are technically all that you need. It's possible to prove everything in propositional logic given the rules we've already discussed. However, as we've already seen, these proofs can get quite long and complicated. We can cut down on the length and difficulty of proofs if we have more inference rules. For all future exams, you will be allowed to use any type of inference rule we've mentioned in class.

Disjunctive Syllogism (DS)

$P \vee Q, \neg P \vdash Q$
$P \vee Q, \neg Q \vdash P$

Given a disjunction and the negation of one of the disjunct, infer that the other disjunct holds.

Example: Use Disjunctive Syllogism in the following proof:

$$P \vee Q, \neg(P \to Q) \vdash P$$

Double Negation Introduction (DN Intro)

$P \vdash \neg\neg P$

Given a formula, infer that its double-negation holds.

Example: Use Double Negation Introduction in the following proof:

$$P \vee \neg Q, R \wedge Q \vdash P \wedge R$$

Modus Tollendo Tollens (MTT)

$P \to Q, \neg Q, \vdash \neg P$

Given a conditional and the negation of its consequent, infer that the negation of the antecedent holds.

Example: Use Modus Tollendo Tollens in the following proof:

$$P \to (Q \leftrightarrow R), Q \leftrightarrow \neg R \vdash \neg P$$

Disjunction Conditional ($\vee \to$)

$P \vee Q \vdash (\neg P) \to Q$
$P \vee Q \vdash (\neg Q) \to P$

Given a disjunction, infer a conditional with the negation of one disjunct as the antecedent, and the other disjunct as the consequent.

Example: Use Disjunction Conditional in the following proof:

$$P \vee \neg(Q \vee R), Q \vdash P$$

DeMorgan Laws (DeM)

$\neg(P \wedge Q) \vdash (\neg P) \vee (\neg Q)$
$\neg(P \vee Q) \vdash (\neg P) \wedge (\neg Q)$
$(\neg P) \vee (\neg Q) \vdash \neg(P \wedge Q)$
$(\neg P) \wedge (\neg Q) \vdash \neg(P \vee Q)$

Example: Use the DeMorgan Laws in the following proof:

$$\neg((P \wedge Q) \rightarrow P), P \vdash Q$$

Practice Problems

Using any rule of inference at your disposal, complete the following proofs:

1. $\neg((P \wedge \neg P) \leftrightarrow Q) \vdash Q$

2. $\vdash ((P \vee Q) \wedge (\neg Q \vee R)) \rightarrow (P \vee R)$

3. $\vdash ((P \wedge Q) \rightarrow R) \rightarrow ((P \rightarrow R) \vee (Q \rightarrow R))$