Slimming Logic Down

Eliminating Connectives

So far, we've worked on building logic up. We've added more and more connectives, and extra rules of inference in order to facilitate proofs. How much is necessary? What is the most impoverished logic we could get away with? For starters, we can get rid of the extra rules of inference—as we learned before, it's possible to prove everything in propositional logic without them. But can we get rid of connectives?

The Elimination of the Biconditional

We want to translate sentences that contain a biconditional into other sentences that don't without changing the truth-values of those sentences. We can do that by constructing a truth-table that has the same values as the biconditional but does not employ that connective. In particular, sentences normally denoted with '$P \leftrightarrow Q$' can be translated to '$(P \rightarrow Q) \wedge (Q \rightarrow P)$.'

P	Q	$P \leftrightarrow Q$	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \wedge (Q \rightarrow P)$
TRUE	TRUE	TRUE	TRUE	TRUE	TRUE
TRUE	FALSE	FALSE	FALSE	TRUE	FALSE
FALSE	TRUE	FALSE	TRUE	FALSE	FALSE
FALSE	FALSE	TRUE	TRUE	TRUE	TRUE

Example: Translate the following sentences in English into sentences of propositional logic *without* the biconditional.

1. John will have cake just in case Sarah does not.

2. The party will be fun if and only if Max and Sophie both come.

3. If the democrats win the presidency, then they will control the Senate if and only if they control the House.

4. Martha and Bertha will major in math just in case the philosophy courses are overenrolled and they pass their exams.

Example: Translate the following pairs sentences in propositional logic to other sentences in propositional logic without the biconditional, and prove that the second follows from the first.

$$(P \leftrightarrow Q) \wedge P \vdash Q$$

$$P \leftrightarrow \neg Q \vdash (P \rightarrow Q) \rightarrow \neg P$$

The Elimination of the Conditional

As with the biconditional, it is possible to express sentences in propositional logic without the conditional. There are a few different translations we could use; as before, what matters is preserving the truth-value of all sentences with conditionals. On one translation, the sentence '$P \rightarrow Q$' can be translated to '$\neg(P \wedge \neg Q)$.'

P	Q	$P \rightarrow Q$	$\neg Q$	$P \wedge \neg Q$	$\neg(P \wedge \neg Q)$
TRUE	TRUE	TRUE	FALSE	FALSE	TRUE
TRUE	FALSE	FALSE	TRUE	TRUE	FALSE
FALSE	TRUE	TRUE	FALSE	FALSE	TRUE
FALSE	FALSE	TRUE	TRUE	FALSE	TRUE

Example: Translate the following sentences in English into sentences of propositional logic *without* the conditional or the biconditional.

1. If Linda does not go to the store, then she will not have milk.

2. Paul will pass his exam only if he studies.

3. Ryan and Wanda will take the bus if the train is late and the bus is not.

4. Fred will wear stripes if and only if Jacob doesn't.

Example: Translate the following pairs sentences in propositional logic to other sentences in propositional logic without the biconditional or conditional, and prove that the second follows from the first.

$$\neg(P \rightarrow Q) \vdash P$$

$$(P \leftrightarrow Q) \wedge P \vdash Q$$

The Elimination of Conjunction

It turns out we can also do without conjunction—we can translate sentences of the form '$P \wedge Q$' into '$\neg(\neg P \vee \neg Q)$'.

P	Q	$P \wedge Q$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$\neg(\neg P \vee \neg Q)$
TRUE	TRUE	TRUE	FALSE	FALSE	FALSE	TRUE
TRUE	FALSE	FALSE	FALSE	TRUE	TRUE	FALSE
FALSE	TRUE	FALSE	TRUE	FALSE	TRUE	FALSE
FALSE	FALSE	FALSE	TRUE	TRUE	TRUE	FALSE

Example: Translate the following sentences in English into sentences of propositional logic *without* the conditional, biconditional or conjunction.

1. Both Sarah and Max did not wear black.

2. If Jill goes to the party, then Jack will not.

3. Sarah, Jane and Leslie all passed the logic exam

4. Paul will have salad just in case Margaret eats chicken.

Example: Translate the following pairs sentences in propositional logic to other sentences in propositional logic without the biconditional, conditional or conjunction and prove that the second follows from the first.

$$P \wedge Q \vdash P$$

$$P \rightarrow Q, P \vdash Q$$

The Elimination of Disjunction

We've only got two logical connectives left—$\neg$ and $\vee$. It is impossible to eliminate either and retain enough expressive power for logic. However, we can reintroduce conjunction and remove disjunction; the connectives $\neg$ and $\wedge$ will work just as well. We can translate '$P \vee Q$' as '$\neg(\neg P \wedge \neg Q)$.'

P	Q	$P \vee Q$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$\neg(\neg P \wedge \neg Q)$
TRUE	TRUE	TRUE	FALSE	FALSE	FALSE	TRUE
TRUE	FALSE	TRUE	FALSE	TRUE	FALSE	TRUE
FALSE	TRUE	TRUE	TRUE	FALSE	FALSE	TRUE
FALSE	FALSE	FALSE	TRUE	TRUE	TRUE	FALSE

Example: Translate the following sentences in English into sentences of propositional logic *without* the conditional, biconditional or disjunction.

1. Either Pete or Rachel went for a jog.

2. If Lucy wins first place, then Alberto will win second.

3. Either Tim or Joseph will bring an umbrella if it is raining.

4. Aramis, Athos and Porthos are excellent duelers.

Example: Translate the following pairs sentences in propositional logic to other sentences in propositional logic without the biconditional, conditional or disjunction, and prove that the second follows from the first.

$$P \vee (Q \wedge \neg Q) \vdash P$$

$$(P \wedge Q) \vee (R \wedge Q) \vdash Q$$

Disjunctive Normal Form

When we reach discussions of meta-logic, it will be useful to speak of a formula's *disjunctive normal form*. A formula is in disjunctive normal form just in case it is a disjunction, where each disjunct is a conjunction of sentence symbols and negations of sentence symbols. For example, $(P \wedge \neg Q) \vee R$ is in disjunctive normal form, but $P \wedge \neg\neg Q$, $P \rightarrow Q$, $(\neg(P \wedge Q)) \vee R$ are not in disjunctive normal form. As it turns out, every well-formed formula can be expressed in disjunctive normal form. We will say that a formula is *satisfiable* (similar to the claim that it is possible for it to be true, depending on the values of the sentence symbols) just in case there is at least one disjunct that does not contain both a sentence symbol and its negation.

Example: Translate the following sentences in propositional logic into their disjunctive normal form. Are they satisfiable?

1. $P \wedge \neg P$

2. $P \vee \neg\neg Q$

3. $P \rightarrow Q$

4. $\neg(P \leftrightarrow Q)$

5. $\neg(\neg P \vee \neg Q)$

6. $P \leftrightarrow Q$

7. $(P \rightarrow Q) \vee (Q \rightarrow P)$

8. $\neg(P \rightarrow Q) \vee R$

A Unary Logic

So far we've been able to reduce the connectives in logic to two—either $\neg$ and $\wedge$, or $\neg$ and $\vee$. It is impossible to reduce propositional logic further to any single connective that we have examined thus far. However, it *is* possible to construct a new connective—one we have not encountered before—that is capable of expressing the entirety of propositional logic. This connective is called the Sheffer Stroke, and is denoted by '|'. Try to figure out what the truth-table of the Sheffer Stroke would be:

P	Q	$P \vert Q$
TRUE	TRUE	
TRUE	FALSE	
FALSE	TRUE	
FALSE	FALSE	

Example: Translate the following sentences in first-order logic to those that *only* employ sentence symbols and the Scheffer Stroke.

1. $P \wedge Q$

2. $P \vee Q$

3. $\neg P$

4. $P \rightarrow Q$

Do we even need one logical connective? Could we construct an interpretable language with no connectives at all?

Practice Problems

Translate the following sentences in propositional logic into those that only employ $\wedge$ and $\neg$:

1. $\neg(P \to \neg Q)$

2. $P \vee (Q \to R)$

Translate the following sentences in propositional logic into those that only employ $\vee$ and $\neg$:

3. $(P \wedge Q) \to R$

4. $(\neg P) \leftrightarrow (\neg Q)$

Translate the following sentences into disjunctive normal form:

5. $(P \vee Q) \to R$

6. $\neg(\neg P \to Q)$

Translate the following sentence in propositional logic into one that only employs $|$.

7. $P \leftrightarrow Q$