The Language of First-Order Logic

A New Language

There are limitations to the expressive power of propositional logic. For example, it seems that the inference 'Gretta has seen my picture, therefore someone has seen my picture' is valid, but there is no way to describe this inference in propositional logic. The goal of First-Order Logic is to expand propositional logic in order to describe inferences involving existential and universal statements.

Previously, the smallest unit of analysis was a sentence-symbol. Now, our atomic sentences will involve predicates and names. Predicates are terms like 'tall,' 'hungry,' and 'person.' We usually represent predicates with capital letters beginning with F. Although predicates often take in a single object, they can also take in collections of objects—'taller than,' 'next to,' etc. It is very important to keep track of the order of objects that fall under predicates: 'Sarah is taller than John' means something very different from 'John is taller than Sarah'!

We usually represent names with lower-case letters beginning with a. So, I might represent 'Martha is hungry' with:

$$F = \text{(1) is hungry}$$
$$a = \text{Martha}$$
$$F(a)$$

Example: Translate the following sentences into First-Order Logic.

1. Both Amy and Mark like cheese.

2. Bertha is between Robin and Lucy

3. If Ahmad comes to the party, then Mark will carpool with Ralph

4. If Rob is taller than Paul and shorter than Wendy, then Wendy is taller than Paul.

First order logic has two more elements: *variables* and *quantifiers*. Usually, we represent variables with lower case letters beginning with x. Grammatically, variables can take the place of names. However, for sentences with variables to mean anything, the variable *must occur within the scope of a quantifier*. The sentence '$F(x)$' doesn't mean anything at all. Does it claim that everything is F? That one thing is F?

The quantifiers are '$\forall x$' and '$\exists x$', which mean 'For all x' and 'There exists an x' respectively. The existential quantifier also is taken to mean 'There is at least one x' and 'For some x.' We use parentheses to specify the scope of the quantifier—so '$\forall x(F(x) \wedge G(x))$' means 'For all x, both $F(x)$ and $G(x)$,' or, more colloquially, 'Everything is both F and G.'

Example: Translate the following sentences into First-Order Logic.

1. Someone cheated on the exam.

2. All humans are mortal.

3. If anyone aced the exam, then Laura did.

4. If everyone is taller than 3 feet, then everyone is taller than 2 feet.

5. All prime numbers greater than 3 are odd.

6. All presidents of the united states are over 35 years old.

The situation becomes a bit more complicated when we use multiple quantifiers, the claim 'There is someone who everyone loves' means something very different than 'Everyone is loved by someone.' The first claims that there is a particular person loved by everyone, while the second claims that everyone is loved by someone or other.

Example: Translate the following sentences into First-Order Logic.

1. Everyone loves everyone.

2. Everyone is loved by someone.

3. If Everyone loves someone, then everyone loves everyone.

4. Only logic students love everyone.

5. If everyone loves everyone, then everyone loves someone.

6. If someone loves everyone who loves someone, then everyone who is loved by someone loves someone.

The language of First-Order Logic also includes a term for identity: =. We usually translate phrases like 'is' and 'the same' into identity. The terms flanking the identity sign are always either names or variables. I might say 'Sarah = Sarah' or 'x = y.' We also sometimes abbreviate '$\neg(x = y)$' with '$x \neq y$'

Example: Translate each of the following into the language of First Order Logic using identity.

1. Hesperus is Phosphorus.

2. Superman can fly and Superman is Clark Kent.

3. If anyone ordered soup, then Cicero is Tully.

Using these resources, we can also use quantifiers to claim express *uniqueness claims*, *numerical claims* and *definite-descriptive claims*. Here are some examples:

1. There is a unique F = $\exists x(Fx \wedge \forall y(Fy \rightarrow (x = y))$
2. There are at least two Fs = $\exists x, y(Fx \wedge Fy \wedge x \neq y)$
3. The F is G = $\exists x(Fx \wedge Gx \wedge \forall y(Fy \rightarrow y = x))$

Example: Translate each of the following into the language of First Order Logic:

1. There is a unique President.

2. There are at least two Senators.

3. There are exactly two Senators

4. There are at least three Congressmen.

5. The Queen of England is old.

6. 2 is the only even prime.

7. The murderer used poison to kill the Duke.

Practice Problems

Translate the following sentences into the language of First-Order Logic.

1. Mitch and Nancy are both cats.

2. Pablo is taller than Martha.

3. Gary fed Willy at 6:00 pm.

4. No dogs are cats.

5. All cats are mammals and all dogs are mammals.

6. All dogs chase some cat or other.

7. If Sally is a cat, then she is not a dog.

8. At least two dogs chase brown cats who were fed by Lilly.

9. Exactly four students got As, and Robert was one of them.

10. If at least two cats chase some dog or other, then exactly one dog chases a cat.