Proofs in First-Order Logic

Simple Rules of Inference

The good news is that most inferences in first-order logic are identical to proofs in propositional logic—we can carry over what we've previously learned. Proofs about conjunction introduction, disjunction elimination, etc. that we talked about before all still apply. And, more good news! Two of the rules of inference involving quantifiers are really straightforward:

Existential Introduction (∃ Intro)
$Fa \vdash \exists x Fx$

Given any formula containing occurrences of a given individual name, infer the result of replacing *some* (and perhaps *all*) occurrences of that name with occurrences of one and the same *new* variable, enclosing the formula in parentheses, and prefixing the existential quantifier that binds the variable.

Universal Elimination (∀ Elim)
$\forall x Fx \vdash Fa$

Given a universal statement whose initial quantifier contains some variable, infer the result of dropping the initial quantifier and replacing *all* occurrences of that variable with occurrences of some name.

Example: Prove each of the following:

1. $\forall xy(Fx \rightarrow Gy), \neg\neg Fa \vdash Gb$

2. $\varnothing \vdash (\forall x Fx) \rightarrow (\exists x Fx)$

Complex Rules of Inference

Unfortunately, it isn't all so easy. How do we eliminate an existential statement? How do we introduce a universal one? The basic thought underlying each inference is the same—we introduce a name for an *arbitrary* object, and then make conclusions about objects generally.

> **Existential Elimination**
> If you have a formula bound by an existential quantifier, begin a subproof by selecting a *new name*, and make the first line of the subproof the result of eliminating the quantifier and replacing *all* occurrences of the relevant variable with the name. When you reach a line in the proof where that name *does not occur*, you may exit the subproof and cite existential elimination.

Examples: Prove each of the following:

3. $\forall x(Fx \rightarrow Gx) \vdash \exists x Fx \rightarrow \exists x Gx$

4. $\exists x(Fx \wedge Gx) \vdash \exists x Fx$

There are two types of Universal Introduction (although, really, one is reducible to the other). The first lets us conclude sentences of the form $\forall x(A \rightarrow B)$ while the second lets us conclude any kind of universal statement whatsoever.

Universal Introduction$_1$ ($\forall$ Intro)

Begin a subproof by introducing a *new name a* and assume that some formula A holds, presumably with occurrences of that name (typically, for example, you could assume Fa). When you reach the line of a subproof B, close the subproof and conclude $\forall x(A \rightarrow B)$ where every occurrence of a is replaced by an occurrence of x. If you have used the variable x before, use a new variable instead.

Example: Prove each of the following:

5. $\forall x(Fx \wedge Gx) \vdash \forall x(Hx \rightarrow Fx)$

6. $\varnothing \vdash \forall x((Fx \wedge Gx) \rightarrow Fx)$

Universal Introduction$_2$ ($\forall$ Intro)

Begin a subproof by introducing a *new name a* and do not assume anything about that name at all. Reach a point in your subproof with a formula A. Close the subproof, and conclude the formula prefixed by a universal quantifier and a *new* variable, replacing all occurrences of a with the variable.

Example: Prove each of the following:

7. $\forall x(Fx \wedge Gx) \vdash \forall x Fx$

8. $\neg \exists x Fx \vdash \forall x \neg Fx$

9.$\forall x \neg(Fx \vee Gx) \vdash \forall x \neg Gx$

The Rules of Identity

The rules for identity are pretty straightforward:

Identity Introduction (= Intro)
$\varnothing \vdash a = a$

Identity Elimination (= Elim)
$a = b, Fa \vdash Fb$

The first of these rules allows you to introduce the claim that something is self identical at any time. The second (which is sometimes called 'Leibniz's Law') lets you infer that an object has a property F if you know that object is identical to another object that is F.

Example: Prove each of the following:

10. $a = b \vdash Fa \to Fb$

11. $a = b, b = c \vdash a = c$

12. $a = b \to Fa \vdash b = a \to Fb$

Practice Problems:

Prove each of the following:

1. $Ka \wedge (Ca \wedge Wa), \forall x(Wx \rightarrow Dx) \vdash \exists x(Kx \wedge Dx)$

2. $\forall x(Fx \rightarrow Gx) \vdash \neg\exists x(Fx \wedge \neg Gx)$

3. $\exists xRxax \vdash \exists x, yRyax$

4. $\exists xFx, \neg\exists xGx, \forall x(Fx \rightarrow (Gx \vee Hx)) \vdash \exists xHx$

5. $Fa, \exists x(x \neq a), \forall x, y((Fx \wedge Fy) \rightarrow x = y) \vdash \exists x\neg Fx$