Introduction to Set Theory

1 The Vocabulary of Set Theory

A *set* is a collection of things.

Sets can be described either:
i) by listing their members (e.g., $A = \{1, 3, 9\}$, $B = \{1, 2, 3, ... \}$)
ii) by stating the property that an object must have in order to be a member of that set (e.g., $A = \{x : x$ is a prime number between 2 and 10$\}$, $B = \{x\colon x$ is a natural number$\}$).

We denote set membership with '$\in$' (e.g., $a \in A$, $a \notin B$).

There is no ordering within sets. Two sets are identical just in case they have the same members. We denote the identity of sets with '=' (e.g., $A = B$, $A \neq B$).

Example: Which of the following sets are identical to each other?
$A = \{1, 3, 5\}$ $B = \{5, 1, 3\}$ $C = \{1, 3, 3, 3, 3, 3, 5\}$ $D = \{3\text{-}2, 3 + 0, 2 + 3\}$ $E = \{1, 2, 5\}$

2 The Empty Set and Singletons

The *empty set* (or null set) is the set that has no members, which we denote with '$\varnothing$'.

Proof: Prove that there is a unique empty set.

Suppose (for reductio) that there are two distinct empty sets A and B. Because $A \neq B$, there is something that is either an element of A but not an element of B, or else an element of B but not an element of A. So, either A or B has at least one element. This contradicts the assumption that A and B are both empty sets. Therefore, there is a unique empty set.

Proof: Prove that $\{\varnothing\}$ is not the empty set.

In order to prove that $\{\varnothing\} \neq \varnothing$, we must either identify an element of $\{\varnothing\}$ that is not an element of $\varnothing$, or else identify an element of $\varnothing$ that is not an element of $\{\varnothing\}$. The empty set itself is an element of $\{\varnothing\}$, but is not an element of $\varnothing$, because the empty set has no elements. Therefore, $\{\varnothing\} \neq \varnothing$.

A *singleton* is a set that contains only one member.

Example: Which of the following sets are singletons?
$A = \{1\}$ $B = \{$U.C. San Diego$\}$ $C = \{1, 2\}$ $D = \varnothing$ $E = \{\{1\}\}$ $F = \{\varnothing\}$

3 Subsets and Power Sets

Set A is a *subset* of set B just in case every member of A is also a member of B. We denote the subset relation with '$\subseteq$' (e.g., $A \subseteq B$).

Note that every set is a subset of itself. The notation for proper subset (i.e., non-identical subset) is '$\subset$'.

Proof: Prove that the empty set is a subset of all sets.

Suppose (for reductio) that there exists some set S that the empty set is not a subset of. By definition, there must be a member of the empty set that is not a member of S. This contradicts the assumption that the empty set has no members. Therefore, the empty set is a subset of all sets.

The *powerset* of a set A is the set of all of A's subsets. We denote the powerset of A with '$\wp(A)$'.

Example: What is the powerset of $\{1, 2\}$? Of $\varnothing$?

4 Intersections and Unions

The *intersection* of sets A and B is the set of members that they have in common. We denote intersection with '$\cap$' (e.g., $A \cap B$). More formally, $x \in A \cap B$ iff both $x \in A$ and $x \in B$.

Example: Let set $A = \{x\colon x \text{ is an even number}\}$, $B = \{x\colon x \text{ is an odd number}\}$, $C = \{x\colon x \text{ is a natural number}\}$. What is the intersection of A with B? Of A with C?

Proof: Prove that the intersection of any set with the empty set is the empty set.

Suppose (for reductio) that there is some set S such that the intersection of S with $\varnothing$ is not the empty set. In this case, their intersection is a set that has at least one member. By definition, the intersection of two sets is the set of members they have in common, so S and the empty set have at least one member in common. This contradicts the assumption that $\varnothing$ has no members. Therefore, the intersection of any set with the empty set is the empty set.

Proof: Prove that A is a subset of B iff $A \cap B = A$.

Suppose that A is a subset of B. By definition, every member of A is also a member of B. Therefore, the members that A and B have in common are all of the members of A, so $A \cap B =$ A. Suppose, instead, that $A \cap B =$ A. Therefore, both A and B contain every member of A, so A is a subset of B. Therefore, A is a subset of B iff $A \cap B = A$.

The *union* of sets A and B is the set containing all members of either A or B (and no others). We denote union with '$\cup$' (e.g., '$A \cup B$'). More formally, $x \in A \cup B$ iff either $x \in A$ or $x \in B$.

Example: What is the union of $\{x\text{: } x \text{ is even}\}$ with $\{x\text{: } x \text{ is odd}\}$? The union of $\{x\text{: } x \text{ is even}\}$ with itself?

Proof: Prove that the union of any set S with the empty set is S.

By definition, the union of a set S with the empty set contains all any only members that are either members of S or else members of the empty set. But the empty set has no members. Therefore, the union of S with $\varnothing$ is S.

5 The Cardinality of Sets

How do we compare the sizes of sets? For finitely large sets, the obvious answer is to count their members. A set with six members is larger than a set with five. How do we know that a set S has five members? We determine whether there is a bijection between the elements of S and $\{1, 2, 3, 4, 5\}$—i.e., just in case each element of S can be mapped to a unique member of $\{1, 2, 3, 4, 5\}$ and vice versa. As it turns out, this applies to infinitely large sets as well. We say two sets have the same cardinality (are the same size) just in case there is a bijection between the sets.

Let A and B be sets. We say that A and B *have the same cardinality* just in case there is a bijection between A and B. We denote this with '$||A|| = ||B||$.'

We say that $||A|| \leqslant ||B||$ just in case there is a one-to-one function from A to B.

We say that $||A|| < ||B||$ just in case $||A|| \leqslant ||B||$ and $||A|| \neq ||B||$.

Example: Do the sets $\{x\text{: } x \text{ is even}\}$ $\{x\text{: } x \text{ is a natural number}\}$ have the same cardinality or different cardinalities?

Practice Problems

1. Compute $\wp(\wp(\emptyset))$

2. Let A and B be sets. Provide counterexamples to each of the following:
   a) If $x \in A$, then $x \subseteq A$.

   b) If $x \in A$, then $x \nsubseteq A$.

   c) If $x \in A \cup B$, then $x \in A$.

   d) If $x \in A$, then $x \in A \cap B$.

3. Either prove or refute the following: For any sets A, B, $\wp(A \cup B) = \wp(A) \cup \wp(B)$