Phil 120: Symbolic Logic Instructor: Samuel Z. Elgin

Relations, Functions and Mathematical Induction

1 Relations

An ordered collection of things is denoted with the brackets '< , >' (e.g., <London, Paris>). We can define this in terms of set theory such that $<x, y> = \{x, \{x, y\}\}$.

The *Cartesian Product* of sets A and B is the set of all ordered pairs in which the first element is a member of A and the second is a member of B, which we denote with 'A x B.' More formally, $A \text{ x } B = \{<x, y>: x \in A \text{ and } y \in B\}$.

Example: Let $A = \{1, 2, 3\}$ and $B = \{\text{red, blue}\}$. What is A x B? A x A?

A set is a *binary relation* just in case it contains only ordered pairs. We say that a binary relation R is *from* sets A to B just in case all ordered pairs within R contain a first element which is an element of A and a second element which is an element of B.

Proof: Prove that any binary relation from A to B is a subset of A x B.

Suppose (for reductio) there were some binary relation R from A to B that was not a subset of A x B. By definition, there is some element of R that is not an element of A x B. Select such an element, and call it $<c, d>$. By the definition of 'from,' c is an element of A and d is an element of B. However, this contradicts the claim that $<c, d>$ is not an element of A x B, which follows from the definition of cartesian products. Therefore, any binary relation from A to B is a subset of A x B.

Let R be a relation:

The *domain* of $R = \{x: \text{there is some } y \text{ such that } <x, y> \in R\}$

The *range* of $R = \{x: \text{there is some } y \text{ such that } <y, x> \in R\}$

The *inverse* of $R = \{<x, y>: <y, x> \in R\}$

Example: Let $A = \{<x, y>: x$ and y are natural numbers between 1 and 10, and x is one less than $y\}$. What is the domain, range and inverse of A?

Let R be a relation on S (i.e., every $<x, y> \in R$ is such that both x and $y \in S$):

R is *reflexive* just in case, for all $x \in S$, $<x, x> \in R$

R is *symmetric* just in case for all $<x, y> \in R$, $<y, x> \in R$

R is *asymmetric* just in case for no elements x and $y \in S$, both $<x, y>$ and $<y, x> \in R$.

R is *antisymmetric* just in case for no distinct elements x and $y \in S$, both $< x, y >$ and $< y, x > \in R$.
R is *transitive* just in case for all $< x, y >$ and $< y, z > \in R$, $< x, z > \in R$.
R is an *equivalence relation* just in case it is reflexive, symmetric and transitive

Example: Which of the following relations are reflexive, symmetric, asymmetric, antisymmetric, transitive and equivalence relations?
$A = \{< x, y >: x$ is a parent of $y\}$
$B = \{< x, y >: x$ is a natural number less than or equal to $y\}$
$C = \{< x, y >: x$ is standing next to $y\}$

2 Functions

A binary relation R is a *function* just in case for all x, y, z, if $< x, y > \in R$ and $< x, z > \in R$, then $y = z$.

Example: Which of the following relations are functions?
$R = \{< x, y >:$ person x has the birthday $y\}$
$S =$ The inverse of R
$T = \{< x, y >: y = x + 1\}$
$U = \{< x, y >: x < y\}$
$V = \{<< x, y >, z >: z = x + y \}$

When the domain of a function f is a subset of A and the range is subset of B, we say that f is a function *from A into B*, which we denote '$f : A \rightarrow B$.' We sometimes denote $< x, y > \in f$ with '$f(x) = y$.'

Function f is *one-to-one* just in case, for any x, y, if $f(x) = f(y)$, then $x = y$.

Example: Which of the following functions are one-to-one?
$A = f(x) = 2x + 5$
$B = \{< x, y >: y$ is the firstborn child of $x \}$
$C = f(x) = \sqrt{x}$

3 Induction

An important proof-theoretic resource is induction. Proof by induction starts by defining a set inductively. The basic idea is that we start with a few basic elements, define a function that generates one element from another, and consider the set of all such elements. Every element of this set is either a basic element, or generated from other elements.

A set A is *closed under* f just in case $f\colon U \rightarrow U$, $A \subseteq U$ and for any object x, $x \in A \rightarrow f(x) \in A$.

Example: Let $f : R \rightarrow R$, where $f(x) = x + 1$. Are the following sets closed under f?
$A = \varnothing$
$B = \{1, 2, 3, \ldots, 100\}$
$C = \{100, 101, 102, \ldots\}$
$D = \{x : x \text{ is a natural number}\}$

Set A is $(B, \{f\})$-*inductive* just in case $B \subseteq A$ and A is closed under f.

Example: Which of the following sets are $(\varnothing, f)$-inductive?
$A = \varnothing$
$B = \{100, 101, 102, \ldots\}$
$C = \{x\colon x \text{ is a natural number}\}$.

Let $C = \cap\{A : A \text{ is } (B, F)\text{-inductive}\}$ where B is a set and F is a set of functions. For example, the set of natural numbers can be defined as $\cap\ \{A\colon A \text{ is } (\{0\}, \{f\})\text{-inductive}\}$.

Example: Define the set of (positive) even numbers inductively:

Suppose we want to prove that every natural number has a property Φ. Because we've defined the set of natural numbers inductively, we can show this by proving two things:

i) $\Phi(0)$
ii) For all x, $\Phi(x) \rightarrow \Phi(x + 1)$

Proof: Prove that every natural number is finitely large.

(Basis Case): Show that the elements of the base set all have the property in question. In this case, 0 is finitely large.

(Inductive Step): We assume that a number k has the property in question (being finitely large). This assumption is called the *inductive hypothesis*. We need to show that $k + 1$ also has this property. This step requires two things: spelling out the inductive hypothesis, and showing that it holds for $k + 1$.

IH: Assume that k is finitely large.

Because $k + 1$ is one more than k, so if k is finitely large, then so is $k + 1$. From the inductive hypothesis, we have that k is finitely large, so $k + 1$ is as well.

Therefore, every natural number is finitely large.

Practice Problems

1. Let $R = \{< a, b >, < a, c >, < b, d >, < b, e >, < c, f >, < c, g >\}$

   a) What is the domain of R?

   b) What is the range of R?

   c) What is the inverse of R?

2. Prove that a relation R is symmetric if and only if R = The inverse of R.

3. For every natural number n, $0 + ... + n = \frac{n(n+1)}{2}$